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Riemann surfaces with a large abelian group of automorphisms

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Abstract

In this paper we classify all Riemann surfaces having a large abelian group of automorphisms, that is having an abelian group of automorphism of order strictly bigger then $4(g-1)$, where g denotes as usual the genus of the Riemann surface.

1. Introduction

In 1890 Schwartz [7] proved that a compact Riemann surface of genus $g \geq 2$ has a finite number of automorphisms.

More precisely, a compact Riemann surface of genus $g \geq 2$ has at most $84(g-1)$ automorphisms: this bound was given by Hurwitz as an application of the now famous Riemann-Hurwitz formula [1] in 1893.

This bound is sharp, as shown by the Klein quartic $\{x^3y + y^3z + z^3x = 0\} \subset \mathbb{P}^2$ studied by Klein [2] in 1879, a genus 3 curve with 168 automorphisms.

However, better bounds can be found with more restrictive assumptions. In 1879 Wiman [8] showed that a cyclic group of automorphisms of a Riemann surface cannot have order larger then $4g+2$. In [5] Nakajima proved the upper bound $4g+4$ for an abelian group of automorphisms.

In this paper we are interested in the problem of finding Riemann surfaces with "many" automorphisms. More precisely, given a compact Riemann surface X of genus $g \geq 2$ and G group of automorphisms of X, G is called *large* if $|G| > 4(g-1)$.

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In Section 2 we recall some known facts about group actions on Riemann surfaces. In particular we recall (cf. [3]) that if G is a large group of automorphisms of X and $Y = X/G$, then Y has genus $g = 0$ and using Riemann-Hurwitz formula one can give some restriction on the number of critical values and on the ramification indices of the projection map $X \to Y$.

In Section 3 we refine the above mentioned restrictions under the assumption that the considered large group of automorphisms is abelian: in Theorem 3.1 we reduce to a list of 3 infinite families and 11 exceptional cases. The proof is based on some results of Nakajima [5] about the stabilizer subgroups of the action of an abelian group of automorphisms.

In Section 4, we construct all compact Riemann surfaces with an abelian large group of automorphisms, showing that all cases in the list in the previous section do occur. In each case we give equations of a singular model of the curve. Finally we prove also that such Riemann surfaces are hyperelliptic for genus $g \geq 8$.

The basic tool used in the proof is the structure theorem for Galois covering, due to Pardini [6].

Notation: From now on we will write Riemann surface X for compact Riemann surface X of genus $g(X) \geq 2$.

2. Riemann surfaces having a large group of automorphisms

In this section we briefly recall some facts about group actions on Riemann surfaces. For further details, see [4].

Let G be a finite group acting holomorphically and effectively on a Riemann surface X, and let $P \in X$ be a fixed point. Then:

(a) the stabilizer subgroup H_P is cyclic;

(b) there are only finitely many points in X having nontrivial stabilizer;

(c) there is a (unique) structure of Riemann surface on the quotient X/G such that the projection $\pi : X \to X/G$ is holomorphic; π has degree $|G|$, and for any point $P \in X$, $\mathrm{mult}_P(\pi) = |H_P|$.

Moreover, assume that H_P is nontrivial and let g be a generator of the stabilizer subgroup H_P. Then there is a local coordinate z on X centered at P such that $g(z) = \lambda z$, where λ is a primitive $|H_P|^{th}$ root of unity.

It follows from above that the quotient map $\pi : X \to X/G$ is a $|G|$-sheeted branched cover of X/G: such a map is called Galois cover (with Galois group G).

Let $Q_1, \ldots, Q_s$ be the critical values of π, and let us associate to each Q_i its *ramification index* $v_i = |H_P|$, where H_P is the stabilizer subgroup of any $P \in \pi^{-1}(Q_i)$. Note that the ramification index of a critical value is well-defined since if P and P' are points of X with $\pi(P) = \pi(P')$, then their stabilizers H_P and $H_{P'}$ are conjugated.

In this special case the Riemann-Hurwitz formula can be written as follows:

$$2g(X) - 2 = |G|\left(2g(X/G) - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{v_i}\right)\right).$$

Applying this formula one can prove the following theorem, due to Kulkarni [3].

We recall that a subgroup G of $Aut(X)$ is said to be *large* if:

$$|G| > 4(g(X) - 1).$$

Theorem 2.1 (Kulkarni [3])

Let X be a Riemann surface of genus $g(X) \geq 2$, let G be a large group of automorphisms of X and consider the quotient surface $Y := X/G$.

Then $g(Y) = 0$ and the projection map $\pi : X \to Y$ has either 3 or 4 critical values. At most the following possibilities for the branching indices can occur.

A) Four critical values

a) One infinite family: $\{2, 2, 2, n\}$ $3 \leq n$
b) Three exceptional cases: $\{2, 2, 3, a\}$ $3 \leq a \leq 5$.

B) Three critical values

a) One infinite family depending on 2 parameters: $\{2, m, n\}$ $3 \leq m \leq n$ *and if $m = 3$ then $n \geq 7$, if $m = 4$ then $n \geq 5$.*

b) Five infinite families:
i) $\{3, a, n\}$ $3 \leq a \leq 6$, $a \leq n$; *if $a = 3$ then $n \geq 4$*
ii) $\{4, 4, n\}$ $4 \leq n$.

c) Exceptional cases:
i) $\{3, 7, a\}$ $7 \leq a \leq 41$
ii) $\{3, 8, a\}$ $8 \leq a \leq 23$
iii) $\{3, 9, a\}$ $9 \leq a \leq 17$
iv) $\{3, 10, a\}$ $10 \leq a \leq 14$
v) $\{3, 11, a\}$ $11 \leq a \leq 13$
vi) $\{4, 5, a\}$ $5 \leq a \leq 19$
vii) $\{4, 6, a\}$ $6 \leq a \leq 11$
viii) $\{4, 7, a\}$ $7 \leq a \leq 9$
ix) $\{5, 5, a\}$ $5 \leq a \leq 9$
x) $\{5, 6, a\}$ $6 \leq a \leq 7$.

In the table above we write the ramification indices of the critical values as a list: therefore, for example, $\{2, 2, 2, n\}$ means "4 critical values, 3 of index 2 and the fourth of index n", and $\{2, m, n\}$ means "3 critical values, one of index 2, one of index m and the third of index n".

So we have seven infinite families (one depending on two parameters) and 102 exceptional cases.

3. The abelian case

In this section we assume G abelian.

Let then X and Y be Riemann surfaces and let $\pi : X \to Y$ be a finite G-abelian cover, i.e. a Galois cover with finite abelian Galois group G. For a point $Q \in Y$ fix a point $P \in \pi^{-1}(Q)$, then the stabilizer subgroup of Q

$$H_Q = \{\sigma \in G | \sigma P = P\}$$

is well-defined, since two subgroups of an abelian group are conjugated if and only if they coincide.

Assuming G abelian not all cases of Kulkarni's list do occur, but we have the following:

Theorem 3.1

Let X be a Riemann surface and G a large abelian group of automorphisms of X. Then at most the following possibilities for the branching indices occur:

A) Four critical values

$G \cong \mathbb{Z}_6$	$\{2,2,3,3\}$.

B) Three critical values

a) Three infinite families

i)	$G \cong \mathbb{Z}_{4g+2}$	$\{2, 2g+1, 4g+2\}$
ii)	$G \cong \mathbb{Z}_{4g}$	$\{2, 4g, 4g\}$
iii)	$G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2g+2}$	$\{2, 2g+2, 2g+2\}$.

b) Ten exceptional cases.

i)	$G \cong \mathbb{Z}_{12}$	$\{3,4,12\}$
ii)	$G \cong \mathbb{Z}_{15}$	$\{3,5,15\}$
iii)	$G \cong \mathbb{Z}_6$	$\{3,6,6\}$
iv)	$G \cong \mathbb{Z}_{21}$	$\{3,7,21\}$
v)	$G \cong \mathbb{Z}_9$	$\{3,9,9\}$
vi)	$G \cong \mathbb{Z}_5$	$\{5,5,5\}$
vii)	$G \cong \mathbb{Z}_3 \times \mathbb{Z}_9$	$\{3,9,9\}$
viii)	$G \cong \mathbb{Z}_3 \times \mathbb{Z}_6$	$\{3,6,6\}$
ix)	$G \cong \mathbb{Z}_5 \times \mathbb{Z}_5$	$\{5,5,5\}$
x)	$G \cong \mathbb{Z}_4 \times \mathbb{Z}_4$	$\{4,4,4\}$.

We note that we are left with only three infinite families and 11 exceptional cases.

In order to prove the previous theorem we use the following result due to Nakajima [5].

Theorem 3.2

Let $X \xrightarrow{\pi} Y$ be a finite G-abelian cover and let $T_1, \ldots, T_s \in Y$ be the critical values of π. For all $j_0 \in \{1, \ldots, s\}$ we denote by G_{j_0} the subgroup of G generated by all the stabilizers subgroup $H_{T_1}, \ldots, H_{T_s}$ except $H_{T_{j_0}}$.

Then, if $g(Y) = 0$ we have $G = G_{j_0}$.

Consequently, if $e_{T_j} := |H_{T_j}|$ and $n_{j_0} := \prod_{j \neq j_0} e_{T_j}$ then n_{j_0} is a multiple of $|G|$.

From Nakajima's theorem it follows easily the following:

Corollary 3.3

If G is cyclic, then the least common multiple of any $s-1$ branching indices equals the order of G.

Applying these results we can exclude many cases from the original Kulkarni's list and prove Theorem 3.1.

Proof of 3.1. We start by considering the cases in Kulkarni's list with 4 critical values.

a) $\{2,2,2,n\}$, $\quad 3 \le n$.

By Theorem 3.2 G must be generated by elements of order 2, contradicting the fact that it contains a cyclic subgroup of order $n \ge 3$. Therefore this case does not occur.

b) $\{2,2,3,a\}$, $\quad 3 \le a \le 5$.

Let us start with the case $a = 3$. In this case clearly $2\big||G|$ and $3\big||G| \Rightarrow 6\big||G|$. Moreover by Theorem 3.2 $|G|\big|12$ and $|G|\big|18 \Rightarrow |G|\big|6$. Then $G \cong \mathbb{Z}_6$.

The same argument for $a = 4$ gives $12\big||G|$ and $|G|\big|16$, and for $a = 5$ gives $30\big||G|$ and $|G|\big|12$, therefore both cases do not occur.

We are left with the cases with 3 critical values.

a) $\{2,m,n\}$, $\quad 3 \le m \le n$.

In this case $2\big||G|$, $m\big||G|$ and $n\big||G|$. By Theorem 3.2, $|G|\big|2m$, $|G|\big|2n$ and $|G|\big|mn$. It follows that either $|G| = n = m$ or $|G| = n = 2m$ or $|G| = 2n = 2m$.

i) $|G| = n = 2m$. In this case $m \ne 3$ (by the condition $m = 3 \Rightarrow n \ge 7$). Moreover (by $|G| = n$) G is cyclic and therefore we can use Corollary 3.3 and conclude that m is odd. By Riemann-Hurwitz formula $m = 2g_X + 1$ and we have group $G \cong \mathbb{Z}_{4g_X+2}$ and ramification indices $\{2, 2g_X + 1, 4g_X + 2\}$.

ii) $|G| = 2n = 2m$. In this case $|G|\big|nm \Rightarrow m$ even. By Riemann-Hurwitz formula $m = 2(g_X + 1)$, $m \ne 4$ (by the condition $m = 4 \Rightarrow n \ge 5$). Moreover, by Corollary 3.3, G cannot be cyclic and therefore $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2g_X+2}$ and the ramification indices are $\{2, 2g_X + 2, 2g_X + 2\}$.

iii) $|G| = n = m$. Arguing as in the previous cases G is cyclic and $4 \ne m = 4g_X$. Therefore $G \cong \mathbb{Z}_{4g_X}$ and the ramification indices are $\{2, 4g_X, 4g_X\}$.

b) $\{3,a,n\}$, $\quad 3 \le a \le 6$ and $\{4,4,n\}$.

If $a = 3$, Kulkarni's list imposes $n \ge 4$. By $n\big||G|\big|3a = 9$, we conclude $n = |G| = 9$ and therefore the group is cyclic and we can apply Corollary 3.3 and get a contradiction.

If $a = 4$ or $a = 5$ we have $3a\big||G|\big|3a$ and therefore clearly $G \cong \mathbb{Z}_{3a}$ and by Theorem 3.2 we have ramification indices $\{3, a, 3a\}$.
If $a = 6$ we have $6\big||G|\big|18$. If $|G| = 6$, $G \cong \mathbb{Z}_6$ and the ramification indices are $\{3, 6, 6\}$. Otherwise $|G| = 18$ and (since by Corollary 3.3 G is not cyclic) the only possibility is $G \cong \mathbb{Z}_3 \times \mathbb{Z}_6$ and ramification indices $\{3, 6, 6\}$.

For the case $\{4,4,n\}$, $n \ge 4$ we first note that by Theorem 3.2 G is generated by two cyclic subgroups of order 4 and therefore no element of G has higher order. In particular $n = 4$. We have $4\big||G|\big|16$. If $|G| = 4$ by Riemann-Hurwitz formula

$$2g_X - 2 = 4\left(1 - \frac{3}{4}\right) = 1,$$

a contradiction.

Then $|G| = 8$ or 16 and G contains three cyclic subgroups of order 4 such that each pair of them generates G: the only possibility is $G \cong \mathbb{Z}_4 \times \mathbb{Z}_4$.

Note that we have already found all the cases in our list except the 5 "exceptional" that we have labeled as iv), v), vi), vii) and ix). It remains to check the last 99 exceptional cases (case "c)") of Kulkarni's list and show that (for G abelian) only these 5 are possible: this can be proved with the same arguments we have already used in this proof, and we leave the details to the reader. $\square$

4. Constructing curves with large abelian group of automorphism: the main theorem

In the previous section we have restricted Kulkarni's list in the abelian case to three infinite families and 11 exceptional cases; in this section we show that all these cases do occur, constructing the curves.

Theorem 4.1

The Riemann surfaces with a large abelian group of automorphisms and 4 critical values are desingularisation of the 1−parameter family of curves of genus 2 in the weighted projective space $\mathbb{P}(2,1,1)$ of equation

$$y^6 = x^3z^3(x-z)^2(x-\lambda z)^4$$

where y is the coordinate of weight 2 and λ varies in $\mathbb{C}\setminus\{0,1,-1\}$.

The group of these curves is isomorphic to $\mathbb{Z}_6$, and the ramification indices are $\{2,2,3,3\}$.

The other Riemann surfaces with a large abelian group of automorphisms are listed in the following table:

1	$\mathbb{Z}_{4g+2}$	$2, 2g+1, 4g+2$	$g \geq 2$	$y^{4g+2} = x^{2g+1}(x-z)^2z^{2g-1}$
2	$\mathbb{Z}_{4g}$	$2, 4g, 4g$	$g \geq 2$	$y^{4g} = x^{2g}(x-z)z^{2g-1}$
3	$\mathbb{Z}_{12}$	$3, 4, 12$	3	$y^{12} = x^4(x-z)^3z^5$
4	$\mathbb{Z}_{15}$	$3, 5, 15$	4	$y^{15} = x^5(x-z)^3z^7$
5	$\mathbb{Z}_6$	$3, 6, 6$	2	$y^6 = x^4(x-z)z$
6	$\mathbb{Z}_{21}$	$3, 7, 21$	6	$y^{21} = x^7(x-z)^3z^{11}$
7	$\mathbb{Z}_9$	$3, 9, 9$	3	$y^9 = x^3(x-z)^5z$
8	$\mathbb{Z}_5$	$5, 5, 5$	2	$y^5 = x(x-z)z^3$
9	$\mathbb{Z}_2 \times \mathbb{Z}_{2g+2}$	$2, 2g+2, 2g+2$	$g \geq 2$	$y^2 = xz$ $w^{2g+2} = (x-z)z^{2g+1}$
10	$\mathbb{Z}_3 \times \mathbb{Z}_9$	$3, 9, 9$	7	$y^3 = xz^2$ $w^9 = (x-z)z^8$
11	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$3, 6, 6$	4	$y^3 = xz^2$ $w^6 = (x-z)z^5$
12	$\mathbb{Z}_5 \times \mathbb{Z}_5$	$5, 5, 5$	6	$y^5 = xz^4$ $w^5 = (x-z)z^4$
13	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$4, 4, 4$	3	$y^4 = xz^3$ $w^4 = (x-z)z^3$

In the second column we wrote the group G, in the third the three ramification indices, in the fourth the genus of the curve, in the last the equations of a birational model of the curve in $\mathbb{P}^2$ or in $\mathbb{P}^3$: in all cases the coordinates are chosen so that the map π is induced by the projection given by the coordinates (x,z).

Note that there are two cases in the above list such that the given group admits a subgroup that is still large: the families "1" and "9", in the case $g = 2$, admit a

subgroup of index 2 that is still large. Looking at the list one easily concludes that these are respectively the cases "8" and "5". So, if one is only interested in the list of Riemann surfaces having a large group of automorphisms, one can drop the cases 5 and 8.

To prove the theorem we use a result of Pardini [6] that classifies abelian covers; we will briefly state her (much more general) result only in the special case of Riemann surfaces.

Let X, Y be Riemann surfaces and let $\pi : X \to Y$ be a finite abelian G-cover.

The direct image sheaf $\pi_*(\mathcal{O}_X)$ splits as a direct sum of line bundles as follows:

$$\pi_*(\mathcal{O}_X) = \bigoplus_{\chi \in G^*} L_\chi^{-1} = \mathcal{O}_X \oplus \left(\bigoplus_{\chi \in G^* \setminus \{1\}} L_\chi^{-1} \right),$$

where $G^* = \mathrm{Hom}(G, \mathbb{C}^*)$ denotes the group of characters of G and G acts on L_χ^{-1} via the character χ.

Let $Q_1, \ldots, Q_s$ be the critical values of π, and let us associate to each critical value Q_i its ramification index e_i, its stabilizer subgroup H_i and a generator ψ_i of H_i^*.

Let $\chi_1, \ldots, \chi_n \in G^*$ be such that G^* is the direct sum of the cyclic subgroups generated by the χ_j's, let d_j be the order of χ_j and write (for the sake of simplicity) L_j for L_{χ_j}.

To each pair Q_i, χ_j we associate $r_{i,j}$ such that

$$(\chi_j)_{|H_i} = \psi_i^{r_{i,j}} \qquad 0 \le r_{i,j} < e_i.$$

We have the following:

Theorem 4.2 (Pardini [6])

Let $\pi \colon X \to Y$ be a finite abelian G-cover of Riemann surfaces.
Then we have the following set of linear equivalences:

$$d_j L_j \equiv \sum_{i=1}^{s} \frac{d_j r_{i,j}}{e_i} Q_i \qquad j = 1, \ldots, n.$$

Conversely, for assigned G, Q_i, H_i, ψ_i, $i = 1, \ldots, s$ to each set of line bundles L_j, $j = 1, \ldots, n$, satisfying the previous linear equivalences there corresponds a unique, up to isomorphism, abelian G-cover of Y branched on $Q_1, \ldots, Q_s$ and such that H_i is the stabilizer subgroup of Q_i and ψ_i the corresponding character.

Proof of 4.1. Let us first write the linear equivalences of Theorem 4.2 in the case of cyclic covers of $\mathbb{P}^1$. In this case there is only one linear equivalence.

Let $Q_1, \ldots, Q_s \in \mathbb{P}^1$ be the critical values, $H_1, \ldots, H_s$ and $e_1, \ldots, e_s$ the respective stabilizer subgroups and ramification indices.

For every generator χ of G^* there is a generator g of G such that

$$\chi(g) = e^{(2\pi i)/n},$$

where n is the order of G. H_i is generated by $g^{\frac{n}{e_i}}$.

For every generator ψ_i of H_i^* there is an α_i with $(\alpha_i, e_i) = 1$ (so that $g^{\alpha_i \frac{n}{e_i}}$ generates H_i) such that

$$\psi_i\Big(g^{\alpha_i n/e_i}\Big) = e^{(2\pi i)/e_i}.$$

It follows that

$$\chi_{|H_i}\Big(g^{\alpha_i n/e_i}\Big) = e^{(2\pi i)/n\cdot\alpha_i\cdot n/e_i} = e^{(2\pi i)/e_i\cdot\alpha_i}.$$

We have then to solve the linear equivalence

$$nL \equiv \sum_{i=1}^{s} \frac{n}{e_i}\alpha_i Q_i$$

for $(\alpha_i, e_i) = 1, \quad 0 \le \alpha_i < e_i$.

Since two divisors on $\mathbb{P}^1$ are linearly equivalent if and only if they have the same degree, we only have to find solutions α_i of the congruence

$$\sum_{i=1}^{s} \frac{n}{e_i}\alpha_i \equiv 0 \pmod{n}$$

with $(\alpha_i, e_i) = 1$, $0 \le \alpha_i < e_i$.

We can now use Theorem 4.2 to construct all the pairs X, G in the list of Theorem 3.1 with G cyclic.

Consider the case:

$$G \cong \mathbb{Z}_{4g+2} \qquad \{2, 2g+1, 4g+2\}.$$

Let $Q_1,\ Q_2,\ Q_3$ be the critical values and $H_1,\ H_2,\ H_3$ the respective stabilizers.

By Theorem 3.2 G is generated by each pair of stabilizers; so $G \cong H_1 \times H_2$ with $H_1 \cong \mathbb{Z}_2$, $H_2 \cong \mathbb{Z}_{2g+1}$.

We have also an isomorphism

$$\begin{array}{ccc} G^* & \xrightarrow{\cong} & H_1^* \times H_2^* \\ \chi & \longmapsto & (\chi_{|H_1}, \chi_{|H_2}). \end{array}$$

In particular for each pair (ψ_1, ψ_2) such that $\langle\psi_1\rangle = H_1^*$ and $\langle\psi_2\rangle = H_2^*$ there is a unique generator χ of G^* such that

$$\chi_{|H_1} = \psi_1 \quad \chi_{|H_2} = \psi_2.$$

The linear equivalence to be solved is then:

$$(4g+2)L \equiv (2g+1)Q_1 + 2Q_2 + \alpha Q_3$$

with $0 < \alpha < 4g+2, \quad (\alpha, 4g+2) = 1$.

The only solution is $\deg L = 1$, $\alpha = 2g-1$. This shows the existence of this case.

In fact this method gives these curves explicitely, as desingularisation of the curves in $\mathbb{P}^2$ of equation

$$y^{4g+2} = x^{2g+1}(x-z)^2 z^{2g-1}.$$

The large group of automorphisms of these curves is isomorphic to $\mathbb{Z}_{4g+2}$, generated by the automorphism induced on the curve by the following automorphism of $\mathbb{P}^2$ of order $4g+2$:

$$(x, y, z) \longmapsto (x, e^{(2\pi i)/(4g+2)}y, z).$$

The projection to the quotient π is induced by the projection of center $(0, 1, 0)$, the coordinates (x, z) of the 3 critical values of π are $Q_1 = (0, 1)$, $Q_2 = (1, 1)$ and $Q_3 = (1, 0)$, of respective ramification index $2, 2g+1$ and $4g+2$.

The other cases with cyclic group and 3 critical values are similar, and we leave the details to the reader.

The one with 4 critical values is slightly different:

$$G \cong \mathbb{Z}_6 \qquad \{2, 2, 3, 3\}.$$

In this case $G \cong H_1 \times H_3$ and analogously $G^* \cong H_1^* \times H_3^*$.

The linear equivalence to be solved is

$$6L \equiv 3Q_1 + 3Q_2 + 2Q_3 + 2\beta Q_4$$

that has the only solution $\deg L = 2$, $\beta = 2$.

The difference is in the explicit construction of the curves, since $\deg L \neq 1$. We have a one-parameter family of curves of genus 2 that are desingularisation of the curves in the weighted projective space $\mathbb{P}(2, 1, 1)$ of equations

$$y^6 = x^3 z^3 (x - z)^2 (x - \lambda z)^4$$

where y is the coordinate of weight 2 and λ varies in $\mathbb{C}$.

We are left with the cases where G is not cyclic. Consider for example

$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2g+2} \qquad \{2, 2g+2, 2g+2\}.$$

Since each pair of stabilizers generates G, $G \cong H_1 \times H_2$ with $H_1 \cong \mathbb{Z}_2$ e $H_2 \cong \mathbb{Z}_{2g+2}$.

Analogously $G^* \cong H_1^* \times H_2^*$: for each pair (ψ_1, ψ_2) with $\langle \psi_1 \rangle = H_1^*$ and $\langle \psi_2 \rangle = H_2^*$, there are two uniquely determined characters χ_1 and χ_2 such that $\langle \chi_1, \chi_2 \rangle = G^*$ and

$$\begin{array}{ll} \chi_{1|H_1} = \psi_1 & \qquad \chi_{2|H_1} = 1 \\ \chi_{1|H_2} = 1 & \qquad \chi_{2|H_2} = \psi_2. \end{array}$$

By Theorem 4.2 we have to solve two linear equivalences:

$$\begin{cases} 2L_1 \equiv Q_1 + \frac{\alpha}{g+1} Q_3 & \text{with} \quad 0 < \alpha < 2g+2, \quad \text{therefore } \alpha \in \{0, g+1\} \\ (2g+2)L_2 \equiv Q_2 + \beta Q_3 & \text{with} \quad 0 < \beta < 2g+2. \end{cases}$$

The only solution is $\deg L_1 = \deg L_2 = 1$, $\alpha = g+1, \beta = 2g+1$. This shows the existence of this case.

These curves are the desingularisation of the curve of equations in $\mathbb{P}^3$

$$\begin{cases} y^2 = xz \\ w^{2g+2} = (x - z) z^{2g+1}. \end{cases}$$

The projection to the quotient is the one induced by the projection $\mathbb{P}^3 \longrightarrow \mathbb{P}^1$ of center $\{x = z = 0\}$. G is generated by

$$\begin{cases} (x, y, z, w) & \longmapsto & (x, -y, z, w) \\ (x, y, z, w) & \longmapsto & (x, y, z, e^{\frac{2\pi i}{2g+2}} w). \end{cases}$$

The same method works in all cases; we leave the details to the reader. □

We have the following

Corollary 4.3

Let X be a Riemann surface and suppose that on X acts an abelian large group of automorphisms. If $g(X) \geq 8$ then X is hyperelliptic.

More precisely all the curves in the Theorem 4.1 are hyperelliptic except the cases 3, 4, 6, 7, 10, 11, 12, 13.

Proof of 4.3. The family of Riemann surfaces with 4 critical values and the cases 5 and 8 have genus 2 and therefore are hyperelliptic.

The involution of $\mathbb{P}^2$ given by $(x, y, z) \mapsto (x, -y, z)$ induces in cases 1 and 2 an involution of the curve with $2g + 2$ fixed points, as the involution of $\mathbb{P}^3$ given by $(x, y, z, w) \mapsto (x, -y, z, w)$ in case 9.

Therefore also the curves in cases 1, 2 and 9 are hyperelliptic, and it remains to show that all the other cases are not hyperelliptic.

In fact, if the hyperelliptic involution τ exists, it belongs to the center of the automorphisms group. It follows that if $\tau \notin G$, there exists G' abelian subgroup of $Aut(X)$ with $|G'| = 2|G| \geq 2(4g-3)$, contradicting (since $g \geq 3$) the upper bound $4g+4$ for the cardinality of an abelian group of automorphism. Therefore, if τ exists, $\tau \in G$.

It is now easy to conclude: the groups in the cases 4, 6, 7, 10 e 12 do not contain any involution, and one can check that the involutions in cases 2, 11 and 13 have not $2g + 2$ fixed points. □

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